

Fused Angles and the Deficiencies of Euler Angles

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Abstract—Just like the well-established Euler angles representation, fused angles are a convenient parameterisation for rotations in three-dimensional Euclidean space. They were developed in the context of balancing bodies, most specifically walking bipedal robots, but have since found wider application due to their useful properties. A comparative analysis between fused angles and Euler angles is presented in this paper, delineating the specific differences between the two representations that make fused angles more suitable for representing orientations in balance-related scenarios. Aspects of comparison include the locations of the singularities, the associated parameter sensitivities, the level of mutual independence of the parameters, and the axisymmetry of the parameters.

I. INTRODUCTION

The fused angles rotation parameterisation was recently introduced in [1]. While it arose from the analysis and control of balancing bodies in 3D and has been used extensively as such [2], it has also since been used for various other purposes, including for example attitude estimation [3] and the modelling of foot orientations and ground contacts [4]. Libraries have also been released in C++ [5] and Matlab [6] that implement a wide variety of conversions and operations involving fused angles and all of the classic ways of representing rotations.

Fused angles aim to provide a robust and geometrically intuitive way of quantifying the amount of rotation that a body has within each of the three major planes, i.e. the xy , yz and xz planes, as illustrated on the left in Fig. 1. This can conceptually be thought of as requiring a notion of how to concurrently quantify the ‘amount of rotation’ a body has about the three principal axes. Furthermore, it is an aim that the three quantified planar rotation values describe the state of balance in an intuitive, problem-relevant and symmetrical way, in particular with respect to the lateral, sagittal and transverse planes. Clearly, quaternions do not satisfy these stated aims as no three quaternion components directly quantify planar angles of rotation. Note that due to the balance-inspired nature of the task, the only required axiom is that there is some clear notion of ‘up’. This is generally the opposite direction to gravity, or along a particular surface normal, and without loss of generality is chosen to be represented by the global z -axis.

Euler angles, illustrated on the right in Fig. 1, may at first seem to satisfy these requirements, being a commonly accepted catch-all solution, but this is not entirely so. They can often enough be the correct choice for a task, such as

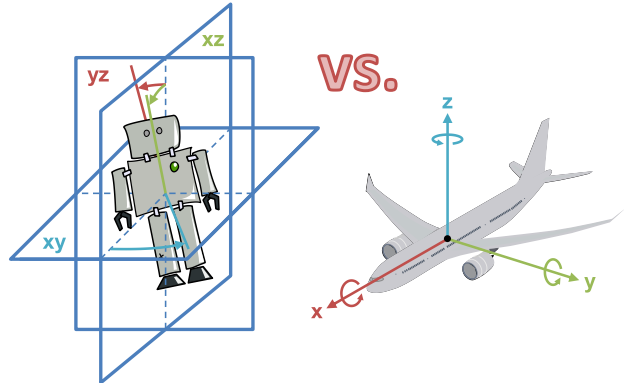


Fig. 1. Comparison of the fundamental principles of fused angles (left) and Euler angles (right). Fused angles quantify the amount of rotation within the three major planes, while Euler angles define three sequential axis rotations, e.g. around the z -axis, y -axis and then the x -axis.

for example for the modelling of gimbals, or a colocated series of joints, but too often they are chosen simply because there does not seem to be a reasonable alternative. This paper critically assesses Euler angles in direct comparison to fused angles, to elucidate the differences that make fused angles the more suitable candidate for quantifying the orientation of a balancing body. This comparative analysis, in addition to the presentation of some noteworthy properties of fused angles, is the main contribution of this paper.

II. RELATED WORK

Based on the aims that were outlined in Section I, one can quickly see why existing rotation representations such as quaternions, rotation matrices, axis-angle pairs [7], rotation vectors [8], and vectorial parameterisations [9] [10], are not appropriate for the task. For example, neither rotation matrices nor quaternions clearly identify components of rotation within the three major planes. A thorough review of all of these representations, and exactly why they are not suitable, can be found in [1]. The only remaining possibly suitable classical rotation representation is Euler angles, which at least at first glance seems to satisfy the specified aims. It is briefly explained in [1] why also this representation does not suffice for the required application, but providing a complete substantiation of this claim is the central topic of this paper.

III. REVIEW OF EULER ANGLES

Euler angles express a rotation as a sequence of three elemental rotations about a predefined set of coordinate axes, in a predefined order. The elemental rotations are either by convention extrinsic about the fixed global x , y and z -axes, or intrinsic about the local x , y and z -axes of the coordinate frame being rotated. For each of these two types, the order of

TABLE I
COMPLETE LIST OF EULER ANGLES AXIS CONVENTIONS

Type	Order of axis rotations
Proper Euler angles	XYX, XZX, YXY, YZY, ZXZ, ZYZ
Tait-Bryan angles	XYZ, XZY, YXZ, YZX, ZXY, ZYX

axis rotations leads to six possible conventions where each axis is used only once, referred to as Tait-Bryan angles, and a further six possible conventions where the first and third axes of rotation are the same, referred to as proper Euler angles. All possible axis conventions are summarised in Table I.

It is easy to see that all extrinsic Euler angles conventions are completely equivalent to the corresponding intrinsic Euler angles conventions, just with the order of rotations reversed. As such, for comparison with fused angles, without loss of generality, intrinsic Euler angles are chosen. It is also desired for the three elemental rotations to be about three different axes, i.e. Tait-Bryan angles, so that the amount of rotation within each of the three major planes can be quantified. Furthermore, fused angles have an initial yaw rotation component about the z-axis, as detailed later in Section IV-B, so to facilitate a sensible comparison, only the intrinsic ZYX and ZXY Euler angles conventions remain as viable candidates. For completeness, both of these Euler angles conventions are presented briefly in the following sections, but unless explicitly otherwise stated, all further references to ‘Euler angles’ will be referring to intrinsic ZYX Euler angles. All arguments and properties that apply to the ZYX representation can also equivalently be reformulated to apply to the ZXY representation, so the choice is arbitrary.

A. Intrinsic ZYX Euler Angles

Let $\{G\}$ denote a global reference frame, and $\{B\}$ be the frame of which the orientation is being expressed. The intrinsic ZYX Euler angles representation consists of the following three sequential rotations: first a rotation by the *Euler yaw* ψ_E about the z-axis, then by the *Euler pitch* θ_E about the new y-axis, and then by the *Euler roll* ϕ_E about the newest x-axis, as illustrated in Fig. 2. The complete Euler angles rotation from $\{G\}$ to $\{B\}$ is then denoted by

$${}^G_B E = (\psi_E, \theta_E, \phi_E) \in (-\pi, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\pi, \pi] \equiv \mathbb{E}. \quad (1)$$

The representation is unique, except at *gimbal lock*, which is when $\theta_E = \pm \frac{\pi}{2}$. The rotation matrix R corresponding to the Euler angles rotation $E = (\psi_E, \theta_E, \phi_E)$ is given by

$$R = R_z(\psi_E)R_y(\theta_E)R_x(\phi_E) \quad (2)$$

$$= \begin{bmatrix} c_{\psi_E}c_{\theta_E} & c_{\psi_E}s_{\theta_E}s_{\phi_E} - s_{\psi_E}c_{\phi_E} & c_{\psi_E}s_{\theta_E}c_{\phi_E} + s_{\psi_E}s_{\phi_E} \\ s_{\psi_E}c_{\theta_E} & s_{\psi_E}s_{\theta_E}s_{\phi_E} + c_{\psi_E}c_{\phi_E} & s_{\psi_E}s_{\theta_E}c_{\phi_E} - c_{\psi_E}s_{\phi_E} \\ -s_{\theta_E} & c_{\theta_E}s_{\phi_E} & c_{\theta_E}c_{\phi_E} \end{bmatrix},$$

where $s_* \equiv \sin(*)$ and $c_* \equiv \cos(*)$. The conversion from Euler angles to quaternion form $q = (w, x, y, z)$ is given by

$$q = (c_{\bar{\phi}_E}c_{\bar{\theta}_E}c_{\bar{\psi}_E} + s_{\bar{\phi}_E}s_{\bar{\theta}_E}s_{\bar{\psi}_E}, s_{\bar{\phi}_E}c_{\bar{\theta}_E}c_{\bar{\psi}_E} - c_{\bar{\phi}_E}s_{\bar{\theta}_E}s_{\bar{\psi}_E}, c_{\bar{\phi}_E}s_{\bar{\theta}_E}c_{\bar{\psi}_E} + s_{\bar{\phi}_E}c_{\bar{\theta}_E}s_{\bar{\psi}_E}, s_{\bar{\phi}_E}s_{\bar{\theta}_E}c_{\bar{\psi}_E} - c_{\bar{\phi}_E}s_{\bar{\theta}_E}s_{\bar{\psi}_E}), \quad (3)$$

where for example $s_{\bar{\theta}_E} = \sin \bar{\theta}_E = \sin(\frac{1}{2}\theta_E)$.

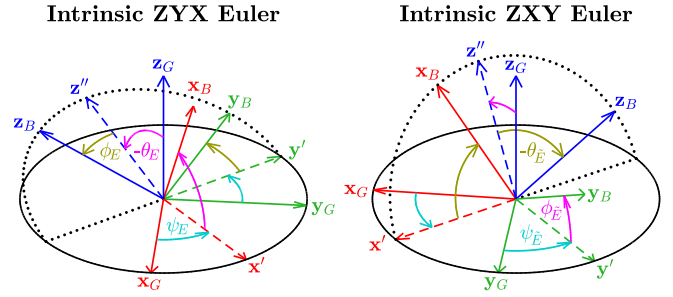


Fig. 2. Diagram of the intrinsic ZYX and ZXY Euler angles parameters. A rotation ${}^G_B R$ is represented by three successive elemental rotations, of angles $(\psi_E, \theta_E, \phi_E)$ about the $\mathbf{z}_G, \mathbf{y}', \mathbf{x}_B$ axes (left), and of angles $(\psi_E, \phi_E, \theta_E)$ about the $\mathbf{z}_G, \mathbf{x}', \mathbf{y}_B$ axes (right), respectively.

B. Intrinsic ZXY Euler Angles

The intrinsic ZXY Euler angles representation consists of the following three sequential rotations: first a rotation by the *ZXY Euler yaw* ψ_E about the z-axis, then by the *ZXY Euler roll* ϕ_E about the new x-axis, and then by the *ZXY Euler pitch* θ_E about the newest y-axis, as illustrated in Fig. 2. The complete ZXY Euler angles rotation is then denoted by

$${}^G_B \tilde{E} = (\psi_E, \phi_E, \theta_E) \in (-\pi, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\pi, \pi] \equiv \tilde{\mathbb{E}}. \quad (4)$$

Relations analogous to (2–3) also hold for ZXY Euler angles. For instance, the quaternion q corresponding to $\tilde{E}$ is

$$q = (c_{\bar{\phi}_E}c_{\bar{\theta}_E}c_{\bar{\psi}_E} - s_{\bar{\phi}_E}s_{\bar{\theta}_E}s_{\bar{\psi}_E}, s_{\bar{\phi}_E}c_{\bar{\theta}_E}c_{\bar{\psi}_E} - c_{\bar{\phi}_E}s_{\bar{\theta}_E}s_{\bar{\psi}_E}, c_{\bar{\phi}_E}s_{\bar{\theta}_E}c_{\bar{\psi}_E} + s_{\bar{\phi}_E}c_{\bar{\theta}_E}s_{\bar{\psi}_E}, c_{\bar{\phi}_E}c_{\bar{\theta}_E}c_{\bar{\psi}_E} + s_{\bar{\phi}_E}s_{\bar{\theta}_E}s_{\bar{\psi}_E}). \quad (5)$$

IV. REVIEW OF FUSED ANGLES

We first briefly introduce the intermediate *tilt angles* representation, and then show how the so-called *tilt rotation component* is reparameterised to yield fused angles. More details on both representations can be found in [1].

A. Tilt Angles

Consider the rotation from a global frame $\{G\}$ to the body-fixed frame $\{B\}$, as shown in Fig. 3. We first construct an intermediate frame $\{A\}$ by rotating $\mathbf{z}_B$ onto $\mathbf{z}_G$ in the most direct way possible within the plane that contains both these vectors. The *fused yaw* ψ is then defined as the angle of the z-rotation from $\{G\}$ to $\{A\}$, and the *tilt angle* α is defined as the angle of the so-called *tilt rotation component* from $\{A\}$ to $\{B\}$. The *tilt axis angle* γ defines the axis in the $\mathbf{x}_G\mathbf{y}_G$ plane about which the tilt rotation occurs, as shown in Fig. 3. The complete tilt angles rotation is denoted by

$${}^G_B T = (\psi, \gamma, \alpha) \in (-\pi, \pi] \times (-\pi, \pi] \times [0, \pi] \equiv \mathbb{T}. \quad (6)$$

All rotations with zero fused yaw ψ are referred to as *tilt rotations*, and are completely and uniquely defined by (γ, α) .

B. Fused Angles

The fused angles representation also uses the fused yaw ψ to represent the z-component of rotation, but reparameterises the tilt rotation component, as shown in Fig. 4. The signed angles between $\mathbf{z}_G$ and the $\mathbf{y}_B\mathbf{z}_B$ and $\mathbf{x}_B\mathbf{z}_B$ planes, respectively, are defined as the *fused pitch* θ and *fused roll* ϕ .

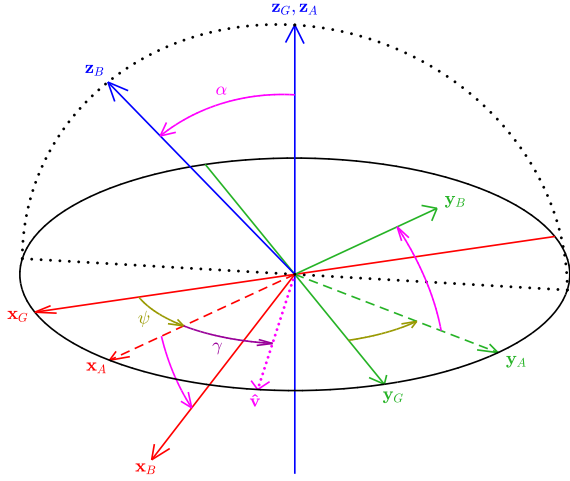


Fig. 3. Diagram of the tilt angles parameters (ψ, γ, α) . A z-rotation by ψ from {G} to {A} is followed by a rotation by α about $\hat{\mathbf{v}}$ from {A} to {B}.

ϕ . A binary *hemisphere* parameter $h \in \{-1, 1\}$ determines which of the two solutions for $\mathbf{z}_B$ are taken. The complete fused angles rotation from {G} to {B} is then denoted by

$${}^G_B F = (\psi, \theta, \phi, h) \in (-\pi, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times \{-1, 1\} \equiv \hat{\mathbb{F}}. \quad (7)$$

Note that $\hat{\mathbb{F}}$ is used because the true domain $\mathbb{F}$ of fused angles is given by the restriction of $\hat{\mathbb{F}}$ by the sine sum criterion

$$\sin^2 \theta + \sin^2 \phi \leq 1 \iff |\theta| + |\phi| \leq \frac{\pi}{2}. \quad (8)$$

Note that ψ , θ and ϕ quantify the amount of rotation within the $\mathbf{xy}$, $\mathbf{xz}$ and $\mathbf{yz}$ major planes of {A}, respectively. Mathematically, if $q = (w, x, y, z) \in \mathbb{Q}$ is the corresponding quaternion, the fused angle parameters are given by

$$\begin{aligned} \psi &= \text{wrap}(2 \text{atan2}(z, w)), & \theta &= \text{asin}(2(wy - xz)), \\ h &= \text{sign}(w^2 + z^2 - \frac{1}{2}), & \phi &= \text{asin}(2(wx + yz)), \end{aligned} \quad (9)$$

where $\text{wrap}(\cdot)$ is a function that wraps an angle to $(-\pi, \pi]$. If R is the corresponding rotation matrix, and R_{ij} denotes the matrix entries, the fused pitch and roll are also given by

$$\theta = \text{asin}(-R_{31}), \quad \phi = \text{asin}(R_{32}). \quad (10)$$

The rotation matrix for $T = (\psi, \gamma, \alpha)$, $F = (\psi, \theta, \phi, h)$ and $\delta \equiv \psi + \gamma$ is given by

$$R = \begin{bmatrix} c_\gamma c_\delta + c_\alpha s_\gamma s_\delta & s_\gamma c_\delta - c_\alpha c_\gamma s_\delta & s_\alpha s_\delta \\ c_\gamma s_\delta - c_\alpha s_\gamma c_\delta & s_\gamma s_\delta + c_\alpha c_\gamma c_\delta & -s_\alpha c_\delta \\ -s_\theta & s_\phi & c_\alpha \end{bmatrix}. \quad (11)$$

V. FUNDAMENTAL PROPERTIES AND RESULTS

The following properties and results are required for the comparative analysis between fused angles and Euler angles.

A. Mathematical Links Between Fused and Euler Angles

Even though the interpretations of the variables are quite different, and the nature of the domains do not correspond, purely mathematically it can be observed that

$$\theta_E = \theta, \quad \phi_E = \phi. \quad (12)$$

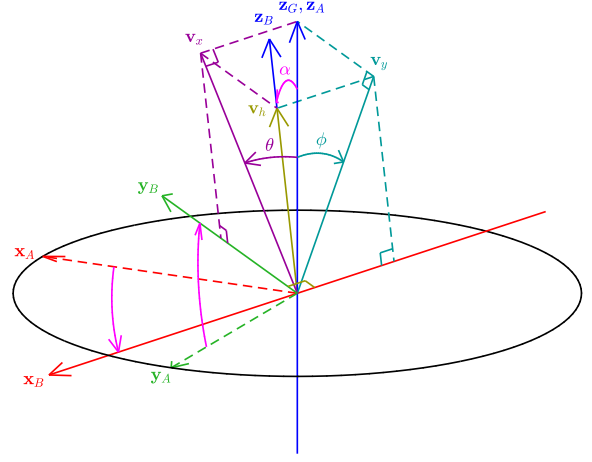


Fig. 4. Diagram of the fused angles parameters (θ, ϕ, h) . The rotation by α from {A} to {B} is reparameterised by the angles θ and ϕ , defined between $\mathbf{z}_G$ and the $\mathbf{y}_B \mathbf{z}_B$ and $\mathbf{x}_B \mathbf{z}_B$ planes, respectively. The hemisphere h is 1 if $\mathbf{v}_h$ is parallel to $\mathbf{z}_B$, and -1 if it is antiparallel.

As such, fused angles can be seen to—with an adaptation of the domains and geometric interpretation—unite the ZYX Euler pitch and ZXY Euler roll with a novel and meaningful concept of yaw, to form a useful representation for rotations.

The following equations relate the ZYX Euler angles, fused angles and tilt angles parameters:

$$\begin{aligned} \phi_E &= \text{atan2}(s_\phi, c_\alpha), & \gamma &= \text{atan2}(s_{\theta_E}, c_{\theta_E} s_{\phi_E}), \\ \phi &= \text{asin}(c_{\theta_E} s_{\phi_E}), & \alpha &= \text{acos}(c_{\theta_E} c_{\phi_E}), \end{aligned} \quad (13)$$

$$h = \text{sign}(c_{\phi_E}) = \begin{cases} 1, & \text{if } |\phi_E| \leq \frac{\pi}{2}, \\ -1, & \text{otherwise,} \end{cases} \quad (14)$$

$$s_\alpha^2 = s_{\theta_E}^2 + s_{\phi_E}^2 - s_{\theta_E}^2 s_{\phi_E}^2. \quad (15)$$

Away from the fused yaw and Euler yaw singularities, the relationship between the two yaws is given by

$$\psi_E = \text{wrap}(\psi + \gamma - \text{atan2}(c_\alpha s_\gamma, c_\gamma)), \quad (16)$$

$$\begin{aligned} \psi &= \text{wrap}(\psi_E - \text{atan2}(s_\theta, s_\phi) + \text{atan2}(s_\theta c_{\phi_E}, s_{\phi_E})) \\ &= \text{wrap}(\psi_E - \text{atan2}(s_{\theta_E}, c_{\theta_E} s_{\phi_E}) + \text{atan2}(s_{\theta_E} c_{\phi_E}, s_{\phi_E})). \end{aligned}$$

B. Effect of Pure Z-Rotations on the Fused Yaw

Unlike for Euler yaw, the composition of any rotation with a pure z-rotation is additive in terms of fused yaw, irrespective of whether the z-rotation is local or global, i.e. applied by post-multiplication or pre-multiplication. If $\Psi(\cdot)$ is the generic operator that returns the fused yaw of a rotation in any representation, then up to angle wrapping

$$\Psi(RR_z(\psi_z)) = \Psi(R_z(\psi_z)R) = \Psi(R) + \psi_z, \quad (17)$$

where $R_z(\psi_z)$ is the rotation matrix corresponding to a pure z-rotation by ψ_z . For global z-rotations, the tilt rotation component also remains unchanged. That is,

$$\begin{aligned} T_z(\psi_z) \circ T(\psi, \gamma, \alpha) &= T(\psi + \psi_z, \gamma, \alpha), \\ F_z(\psi_z) \circ F(\psi, \theta, \phi, h) &= F(\psi + \psi_z, \theta, \phi, h), \end{aligned} \quad (18)$$

where $T(\cdot)$, $F(\cdot)$ is notation that clarifies that the enclosed parameters are tilt angles or fused angles, respectively, and $T_z(\psi_z)$, $F_z(\psi_z)$ correspond to pure z-rotations by ψ_z .

C. Format of Rotation Inverses

The inverses of quaternion rotations and rotation matrices are simply given by:

$$q^{-1} = q^*, \quad R^{-1} = R^T, \quad (19)$$

where q^* is the quaternion conjugate. For Euler angles, the situation is more complicated. For $E = (\psi_E, \theta_E, \phi_E)$,

$$\begin{aligned} E^{-1} &= (\psi_{Einv}, \theta_{Einv}, \phi_{Einv}), \\ \psi_{Einv} &= \text{atan2}(c_{\psi_E} s_{\theta_E} s_{\phi_E} - s_{\psi_E} c_{\phi_E}, c_{\psi_E} c_{\phi_E}), \\ \theta_{Einv} &= -\text{asin}(c_{\psi_E} s_{\theta_E} c_{\phi_E} + s_{\psi_E} s_{\phi_E}), \\ \phi_{Einv} &= \text{atan2}(s_{\psi_E} s_{\theta_E} c_{\phi_E} - c_{\psi_E} s_{\phi_E}, c_{\theta_E} c_{\phi_E}). \end{aligned} \quad (20)$$

For tilt and fused angles, the rotation inverses are given by

$$\begin{aligned} T^{-1} &= (-\psi, \text{wrap}(\psi + \gamma - \pi), \alpha), \\ F^{-1} &= (-\psi, \theta_{inv}, \phi_{inv}, h), \\ \theta_{inv} &= -\text{asin}(s_{\alpha} s_{\psi + \gamma}) = -\text{asin}(c_{\psi} s_{\theta} + s_{\psi} s_{\phi}), \\ \phi_{inv} &= -\text{asin}(s_{\alpha} c_{\psi + \gamma}) = \text{asin}(s_{\psi} s_{\theta} - c_{\psi} s_{\phi}). \end{aligned} \quad (21)$$

It is quite remarkable to note from (21) that

$$\Psi(R^{-1}) = -\Psi(R). \quad (22)$$

This property of fused yaw is referred to as negation through rotation inversion, and is clearly not satisfied by any variant of Euler yaw. Furthermore, for the case of pure tilt rotations, i.e. zero fused yaw, the fused pitch and roll also satisfy the negation through rotation inversion property:

$$\psi = 0 \iff F^{-1} = (0, -\theta, -\phi, h). \quad (23)$$

For zero Euler yaw ψ_E , the expression for the inverse rotation does not simplify as significantly:

$$\begin{aligned} E^{-1} &= (\text{atan2}(s_{\theta_E} s_{\phi_E}, c_{\theta_E}), \\ \psi_E = 0 &\iff \text{asin}(-s_{\theta_E} c_{\phi_E}), \\ &\text{atan2}(-s_{\phi_E}, c_{\theta_E} c_{\phi_E})). \end{aligned} \quad (24)$$

VI. COMPARATIVE ANALYSIS

The fused and Euler angles representations are critically compared in this section. In particular, the many differences between the two representations that make fused angles superior to Euler angles for representing orientations are delineated. The main drawbacks of Euler angles are:

- The proximity of the gimbal lock singularity to normal working ranges, leading to unwanted artefacts due to increased local parameter sensitivities in a widened neighbourhood of the singularity,
- The interdependence of the Euler parameters, leading to an unclear attribution of which parameter encapsulates which major plane of rotation,
- The asymmetry introduced by the use of a definition of yaw that depends on projection, leading to unintuitive non-axisymmetric behaviour of the yaw angle, and
- The fundamental requirement of an order of elemental rotations, leading to non-axisymmetric definitions of pitch and roll that do not correspond in behaviour.

A. Singularities and Local Parameter Sensitivities

It was shown by Stuelpnagel [11] that it is topologically impossible to have a global three-dimensional parameterisation of the rotation group without any singular points. That is, every three-dimensional parameterisation of the rotation space, including both Euler angles and fused angles, must have at least one of the following:

- A rotation that does not have a unique set of parameters,
- A parameter set that does not specify a unique rotation,
- A rotation in the neighbourhood of which the sensitivity of the map from rotations to parameters is unbounded.

The Euler angles representation is singular at gimbal lock, i.e. $\theta_E = \pm \frac{\pi}{2}$. For $\lambda \in \mathbb{R}$, the following equivalences hold:

$$\begin{aligned} (\psi_E, \frac{\pi}{2}, \phi_E) &\equiv (\psi_E - \lambda, \frac{\pi}{2}, \phi_E - \lambda), \\ (\psi_E, -\frac{\pi}{2}, \phi_E) &\equiv (\psi_E - \lambda, -\frac{\pi}{2}, \phi_E + \lambda). \end{aligned} \quad (25)$$

It can be seen that ψ_E, ϕ_E both have essential discontinuities at gimbal lock, and each correspond to type (i) and (iii) singularities. Fused angles only possess a single singularity:

$$\begin{aligned} \text{Singular } \psi &\iff \alpha = \pi \iff w = z = 0 \\ &\iff \theta = \phi = 0 \text{ and } h = -1 \\ &\iff R_{33} \equiv {}^G z_{Bz} \equiv {}^B z_{Gz} = -1. \end{aligned} \quad (26)$$

The so-called fused yaw singularity is also an essential discontinuity, and is of type (i) and (iii), when considering the geometric definition of the parameters, like for Euler angles.

The local state of balance of a body is a function of pitch and roll, but not yaw, as this just determines the heading. Thus, it is critical to compare that fused angles have a single singularity in a single parameter, namely the fused yaw, while Euler angles have two singularities in two parameters, including, very importantly, one that is not yaw. Hence, fused angles can represent local states of balance completely without singularities, while this is not the case for Euler angles. The fused yaw singularity is also ‘maximally far’ from the identity rotation, at 180° , while the two Euler singularities are only 90° away, which is close to, if not in, normal working ranges. In fact, the increased parameter sensitivity of the Euler yaw and roll near gimbal lock has noticeable effects even for tilt rotations of only 65° , as can be seen in Fig. 5. Sudden sensitive changes in Euler yaw and roll occur even when the tilt rotation is actually only a few degrees from being pure pitch—something that is highly problematic. Consequently, the Euler yaw component of a rotation cannot in general be meaningfully removed, as for even moderate tilts this can lead to large z-rotations occurring in the rotation that remains, which should actually only be the contribution of pitch and roll.

B. Mutual Independence of Rotation Parameters

To fulfil the parameterisation aims that were set out in Section I, one necessary condition is that the individual parameters should be as mutually independent as possible, and correspond intuitively to the x, y and z-components of rotation. This is not the case for Euler angles, as is shown by comparison to fused angles in the following subsections.

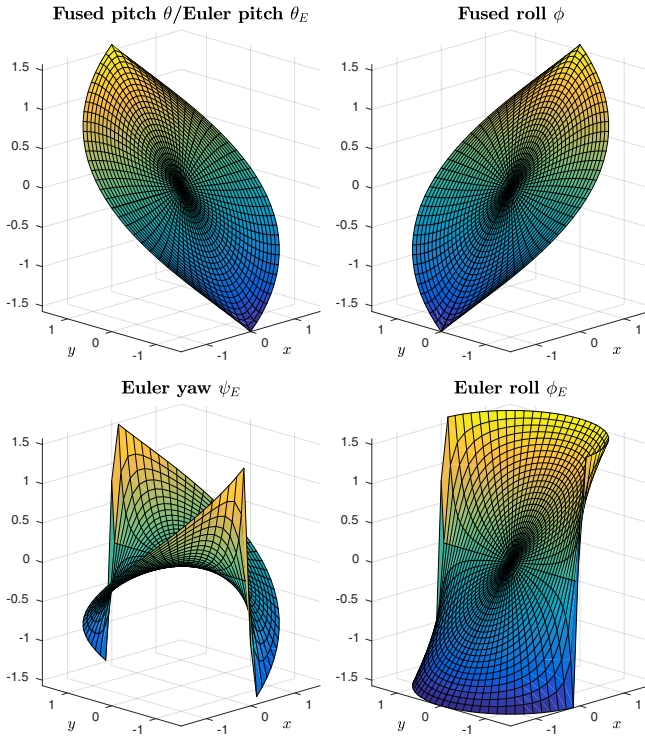


Fig. 5. Plots of the fused and Euler angles parameters when an upright body is rotated about every axis in the xy plane. These are pure tilt rotations as no component of rotation is about the z -axis. The parameters of a rotation by α about the axis $(c_\gamma, s_\gamma, 0)$ are plotted at $(x, y) = (\alpha c_\gamma, \alpha s_\gamma)$. The fused yaw plot is omitted because it is simply perfectly zero everywhere. It can clearly be seen that fused pitch and roll correspond to each other in behaviour, while Euler pitch and roll do not, and that Euler yaw is not axisymmetric, while the fused yaw trivially is, as it is zero everywhere.

Interdependence of Yaw and Roll: After the Euler yaw elemental rotation has been applied, the axis y' of the following Euler pitch rotation (see Fig. 2) always lies in the xy plane. For the Euler roll rotation however, the axis x_B in general has a non-zero z -component, and thus applies a contribution to ‘yaw’ in the intuitive sense. This effect can be seen most clearly at gimbal lock, where Euler yaw and roll become completely interchangeable, as given by (25). Conceptually, part of the total ‘yaw’ of a rotation is always quantified by the ϕ_E parameter instead of ψ_E , meaning that neither parameter cleanly represents the component of rotation that it ideally should. This can also be observed in Fig. 5. Fused angles do not have this kind of interdependence.

Interdependence of Pitch and Roll: As the Euler pitch elemental rotation precedes the Euler roll one, the axis of rotation x_B of the latter is a function of θ_E . This creates a dependency of ϕ_E on θ_E , which results in ϕ_E not completely capturing the intuitive sense of ‘roll’ independently by itself. This can be seen in the bottom row of (2), which is a heading-independent measure of the global up direction, i.e. just like an accelerometer would measure gravity. While the R_{31} entry is a pure function of θ_E , the R_{32} entry is not a function purely of ϕ_E , as would naturally be desired. It can be seen from (11) however, that both these properties hold for fused angles.

Purity of the Axis of Rotation: Euler’s rotation theorem [12] states that every rotation can be expressed as a single

rotation about some vector e . By definition, this vector must lie on the line defined by the (x, y, z) quaternion parameters. Thus, the relative ratios of these quaternion parameters gives insight into the proportions of the rotation that are about each of the corresponding axes. It is known from (9) that

$$\psi = 0 \iff z = 0. \quad (27)$$

This can be interpreted as saying that the fused yaw is zero exactly when there is no component of rotation about the z -axis. This is quite logical, but not the case for Euler yaw.

The yaw components ψ and ψ_E simply rotate the axis of rotation e around the z -axis by half their value. As such, we can inspect the purity of the x and y -axis components by examining just rotations with zero yaw. For fused angles with zero ψ , it can be deduced from (9) that e is on the line $(s_\phi, s_\theta, 0)$. From (3), for Euler angles with zero ψ_E , e is on the line $(s_{\bar{\phi}_E} c_{\bar{\theta}_E}, c_{\bar{\phi}_E} s_{\bar{\theta}_E}, -s_{\bar{\phi}_E} s_{\bar{\theta}_E})$. While for fused angles it can be seen that there is no component of rotation about the x and y -axes exactly when ϕ and θ are zero respectively, for Euler angles, e_y is also zero when $\phi_E = \pi$. This comes about because the e_x and e_y components are mixed expressions of Euler pitch and roll, instead of clean independent expressions like for fused angles, where direct one-to-one associations can be made between $e_x \leftrightarrow \phi$ and $e_y \leftrightarrow \theta$. It is also evident from the $e_z = -s_{\bar{\phi}_E} s_{\bar{\theta}_E}$ term that Euler pitch and roll together contribute a component of rotation about the z -axis, which is unintuitive. In fact, from (5), the ZXY Euler angles expression for e is $(s_{\bar{\phi}_E} c_{\bar{\theta}_E}, c_{\bar{\phi}_E} s_{\bar{\theta}_E}, s_{\bar{\phi}_E} s_{\bar{\theta}_E})$, which has the exact opposite contribution to e_z . As such, as $e_z = 0$, fused angles can conceptually be thought of as being exactly inbetween ZYX and ZXY Euler angles in terms of how the x and y contributions are combined—concurrent and neutral, instead of asymmetrical due to a discrete order of rotations.

Rotation Inverses: The fused yaw satisfies the remarkable negation through rotation inversion property, (22). This property is quite logical, as the component of rotation about the z -axis is negated for the inverse rotation, as can be seen from the inverse quaternion in (19). As a corollary, the inverse of a zero fused yaw rotation also has zero fused yaw. Despite being very natural, neither of these two properties hold for Euler yaw. The inverse equation (20) for Euler yaw actually depends on both Euler pitch and roll, demonstrating that these parameters are interdependent. Even for rotations with zero Euler yaw, (24) shows that all inverse terms are mixed combinations of pitch and roll, including notably the non-zero inverse Euler yaw. By comparison, it can be seen from (23) that for rotations with zero fused yaw, the inverse fused angles rotation resolves trivially into negation through rotation inversion for both fused pitch and roll.

C. Axisymmetry of Yaw

When using the fused angles representation in balance-related applications, by design the z -axis is chosen to point in the direction opposite to gravity. This ensures that the concepts of roll, pitch, and in particular yaw, line up with what one would intuitively expect. The choice of z -axis however still leaves one degree of rotational freedom open

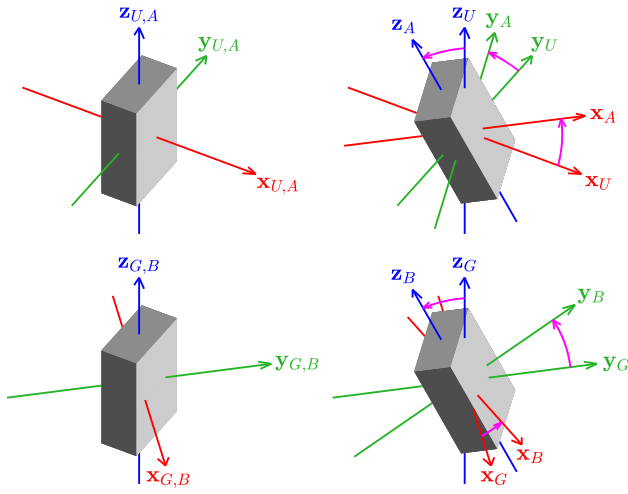


Fig. 6. Definition of frames for the investigation of axisymmetry, where the left side in each row is before rotation, and the right side is after. The same physical rotation is applied in each row. Prior to rotation, $\{U\} = \{A\}$ and $\{G\} = \{B\}$, but $\{U\}$ and $\{G\}$ are fixed to the environment, while $\{A\}$ and $\{B\}$ rotate with the robot. Thus, ${}^U_A R$ and ${}^G_B R$ represent the exact same physical rotation, but with respect to different reference frames, and so are not numerically equal. The axisymmetry of fused yaw asserts that irrespective of the choice of $\{G\}$, the fused yaws of ${}^U_A R$ and ${}^G_B R$ are equal.

for the choice of global x and y-axis. In the context of this paper, the concept of *axisymmetry* refers to the property that one or more rotation parameters are either invariant to this freedom of choice in the axes, or vary in an intuitive rotational manner proportional to the choice. In other words, axisymmetry refers to the notion that the rotation parameters, in order to be self-consistent, should be symmetrical about the unambiguously defined z-axis. This is a relatively natural property to desire, as, for example, the amount of yaw a rotation has should clearly transcend any arbitrary choice of which reference frame to use.

The fused yaw is axisymmetric in the sense that it is invariant to the choice of global x and y-axis. Consider a robot that is upright, and thereby considered mathematically to have an identity orientation $\mathbb{I}_3$ relative to its environment. If the robot undergoes any rotation, the above statement of fused yaw axisymmetry asserts that the fused yaw of this rotation is the same no matter what choice of global x and y-axis was made. This is an important and reassuring property of the fused yaw as, given that the z-axis is unambiguously defined, any concept of yaw about the z-axis should clearly be a property of the actual physical rotation, not a property of some arbitrary choice of virtual reference frame made solely for the purpose of mathematical analysis. It can easily be demonstrated, with virtually any non-degenerate example, that Euler yaw is not axisymmetric, and for different choices of axes can readily produce deviations up to 180° .

Let $\{U\}$ be a global coordinate frame such that $\mathbf{z}_U$ points in the direction opposite to gravity, as required, and suppose that the rotation that is undergone by the robot is given by ${}^U_A R$. This is a fixed physical rotation of the robot relative to its environment, so it should have a unique well-defined fused yaw according to axisymmetry. As the z-axis is fixed, every valid global coordinate system $\{G\}$ that can be used

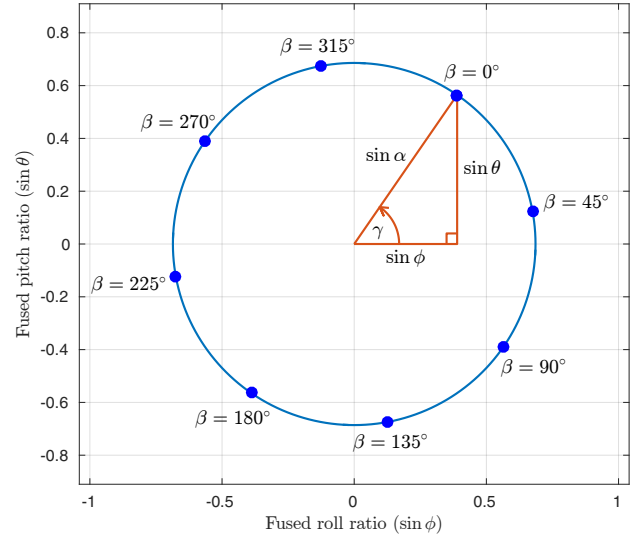


Fig. 7. Locus of the sine ratios $(\sin \phi_\beta, \sin \theta_\beta)$ as β varies, i.e. for all possible choices of x and y-axis, where $(\theta_0, \phi_0) = (0.6, 0.4)$. Refer to (32) for the definitions of θ_β , ϕ_β , θ_0 and ϕ_0 . The triangle demonstrates the decomposition of $\sin \alpha$ into the quadrature sinusoid components $\sin \theta$ and $\sin \phi$, and shows how β can be seen as a negative offset to γ .

as a reference frame to quantify ${}^U_A R$, including $\{U\}$ itself, is a pure z-rotation of $\{U\}$. That is, for some angle β ,

$${}^U_G R = R_z(\beta). \quad (28)$$

Given any choice of $\{G\}$, a frame $\{B\}$ is attached to the robot in such a way that it coincides with $\{G\}$ when the robot is initially upright, and rotates with the robot, as shown in Fig. 6. As such, ${}^U_A R$ and ${}^G_B R$ are simply two different ways of quantifying the exact same rotation, just with a different frame of reference. The rotation ${}^U_A R$ maps $\{G\}$ onto $\{B\}$, so

$${}^U_A R = {}^U_G R {}^G_B R {}^G_U R. \quad (29)$$

Taking the fused yaw of both sides, twice applying (17) as ${}^U_G R$ and ${}^G_U R$ are both pure z-rotations, and using (22), gives

$$\begin{aligned} \Psi({}^U_A R) &= \Psi({}^U_G R {}^G_B R {}^G_U R) \\ &= \Psi({}^U_G R) + \Psi({}^G_B R) + \Psi({}^G_U R^T) \\ &= \Psi({}^U_G R) + \Psi({}^G_B R) - \Psi({}^U_G R) \\ &= \Psi({}^G_B R). \end{aligned}$$

We note that $\Psi({}^U_A R)$ is clearly independent of the choice of $\{G\}$, so $\Psi({}^G_B R)$ must also be. This demonstrates that the fused yaw of the rotation is invariant to the choice of reference x and y-axis, i.e. choice of β , as required.

To show that Euler yaw violates axisymmetry, consider

$${}^U_A R = R_x(\frac{3\pi}{4}), \quad \beta = \frac{\pi}{2}.$$

The Euler yaw of ${}^U_A R$ is clearly zero, but from (28–29),

$$\begin{aligned} {}^G_B R &= {}^G_U R {}^U_A R {}^U_G R \\ &= R_z(-\frac{\pi}{2}) R_x(\frac{3\pi}{4}) R_z(\frac{\pi}{2}) \\ &= E_R(\pi, -\frac{\pi}{4}, \pi), \end{aligned} \quad (30)$$

where $E_R(\cdot)$ denotes the rotation matrix corresponding to the given Euler angles parameters. Thus, the Euler yaw of ${}^G_B R$ is

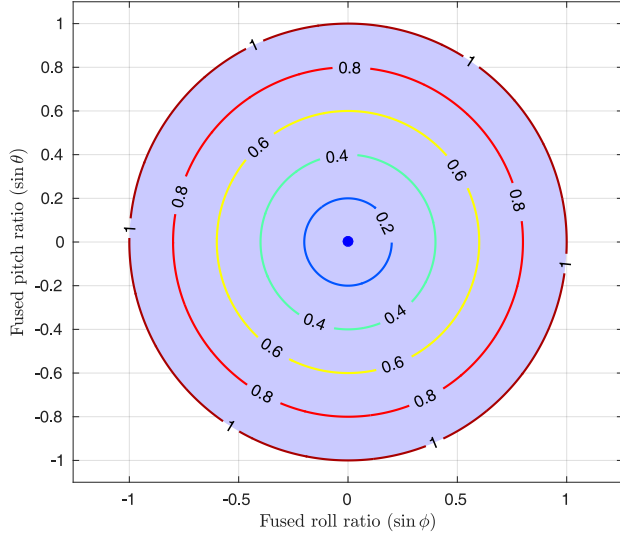


Fig. 8. Level sets of constant $\sin \alpha$ —the sine of the magnitude of the tilt rotation component of a rotation—in the fused pitch ratio $\sin \theta$ vs. fused roll ratio $\sin \phi$ Cartesian space. The shaded region is the valid domain of $(\sin \phi, \sin \theta)$ for the fused angles representation. The purely circular nature of the plot visually illustrates the axisymmetry of fused pitch and roll.

π , which is totally different to that of ${}^U_A R$. This proves that the Euler yaw cannot be axisymmetric. The non-axisymmetry of Euler yaw is visualised in Fig. 11.

D. Axisymmetry of Pitch and Roll

The fused pitch and roll are axisymmetric in the sense that their *sine ratios* $\sin \theta$, $\sin \phi$ circumscribe a uniform circle as a function of the choice of x and y-axis. That is, the locus of $(\sin \phi, \sin \theta)$ over all possible choices of axes is a circle, and this circle is traversed uniformly as the choice varies. As demonstrated later, this is not the case for Euler angles.

The fused pitch θ and fused roll ϕ come together with the hemisphere h to define the tilt rotation component of a rotation. The magnitude of this tilt rotation is given by the tilt angle α , and the relative direction of this tilt rotation is given by the tilt axis angle γ . The angles ϕ and θ can be thought of as a way of ‘splitting up’ the action of α into its orthogonal components. More precisely, the sine ratios $\sin \phi$ and $\sin \theta$ are in fact a decomposition of $\sin \alpha$ into quadrature sinusoid components, as illustrated in Fig. 7 and Fig. 11, and as embodied by

$$\begin{aligned} \sin \alpha &= \sqrt{\sin^2 \phi + \sin^2 \theta}, \\ \gamma &= \text{atan2}(\sin \theta, \sin \phi). \end{aligned} \quad (31)$$

The property of axisymmetry in fused pitch and roll is equivalent to stating that the choice of global x and y-axis simply results in a fixed phase shift to the quadrature components. This suggests that the nature of fused pitch and roll in expressing a rotation is a property of the actual physical rotation, not whatever arbitrary reference frame is chosen to numerically quantify it. It can easily be demonstrated that Euler pitch and roll are not axisymmetric.

To demonstrate the fused pitch and roll axisymmetry mathematically, consider a robot undergoing the same rotation as

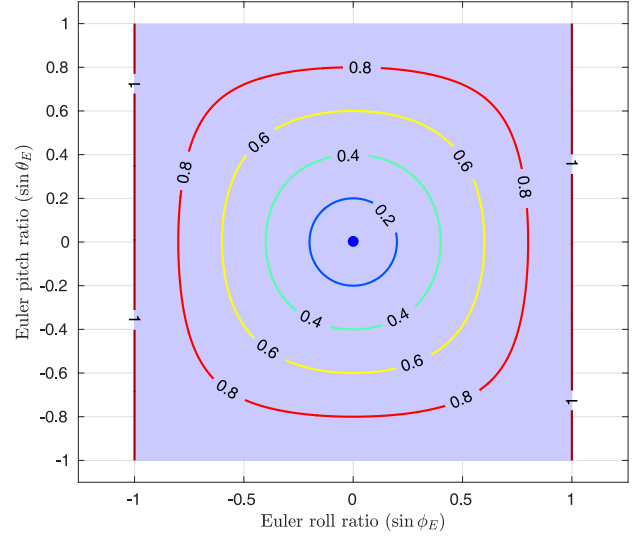


Fig. 9. Level sets of constant $\sin \alpha$ in the Euler pitch ratio $\sin \theta_E$ vs. Euler roll ratio $\sin \phi_E$ Cartesian space.

in Fig. 6. Once again, β embodies the freedom of choice of the x and y-axis. We introduce the notation

$$\begin{aligned} {}^U_A R &= T_R(\psi_0, \gamma_0, \alpha_0) = F_R(\psi_0, \theta_0, \phi_0, h_0), \\ {}^G_B R &= T_R(\psi_\beta, \gamma_\beta, \alpha_\beta) = F_R(\psi_\beta, \theta_\beta, \phi_\beta, h_\beta), \end{aligned} \quad (32)$$

for the tilt and fused angles representations of ${}^U_A R$ and ${}^G_B R$ respectively, where $T_R(\cdot)$ and $F_R(\cdot)$ is notation for the rotation matrices corresponding to the enclosed tilt and fused angles parameters, respectively. Using this notation, the previously established axisymmetry of fused yaw is equivalent to the statement

$$\psi_\beta = \psi_0. \quad (33)$$

Substituting (28) into (30), and applying (11) to ${}^U_A R$ gives

$$\begin{aligned} {}^G_B R &= R_z(-\beta) {}^U_A R R_z(\beta) \\ &= \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ s_\beta s_{\phi_0} - c_\beta s_{\theta_0} & c_\beta s_{\phi_0} + s_\beta s_{\theta_0} & c_{\alpha_0} \end{bmatrix}, \end{aligned} \quad (34)$$

where the ‘ $\cdot$ ’ entries are omitted for brevity. Using (11) to expand ${}^G_B R$, and comparing matrix entries to (34), gives

$$\begin{aligned} -s_{\theta_\beta} &= s_\beta s_{\phi_0} - c_\beta s_{\theta_0}, & c_{\alpha_\beta} &= c_{\alpha_0}, \\ s_{\phi_\beta} &= c_\beta s_{\phi_0} + s_\beta s_{\theta_0}, \end{aligned} \quad (35)$$

This demonstrates the axisymmetry of the tilt angle α ,

$$\alpha_\beta = \alpha_0, \quad (36)$$

and as $h = \text{sign}(c_\alpha)$, also the axisymmetry of h ,

$$h_\beta = h_0. \quad (37)$$

(35) also leads to the matrix equation

$$\begin{bmatrix} \sin \phi_\beta \\ \sin \theta_\beta \end{bmatrix} = \begin{bmatrix} c_\beta & s_\beta \\ -s_\beta & c_\beta \end{bmatrix} \begin{bmatrix} \sin \phi_0 \\ \sin \theta_0 \end{bmatrix}. \quad (38)$$

By identifying the middle matrix as a 2D rotation matrix that rotates clockwise by β , this equation can be seen to be the

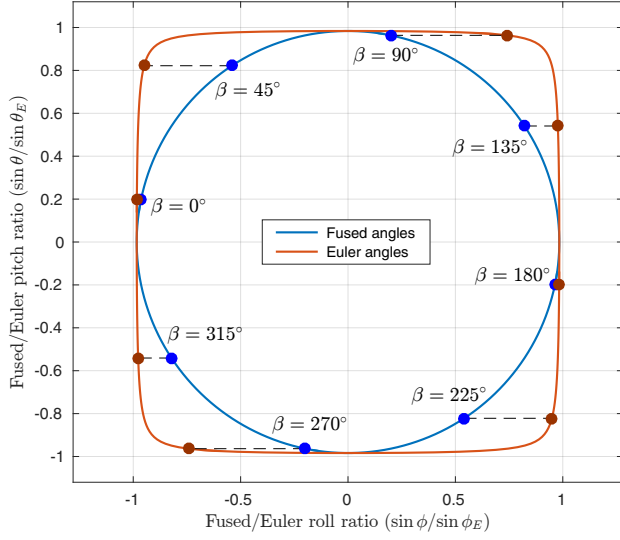


Fig. 10. Loci of the sine ratios $(\sin \phi_\beta, \sin \theta_\beta)$ as β varies, i.e. for all possible choices of x and y-axis, once (inner curve) for fused pitch θ and roll ϕ , and once (outer curve) for Euler pitch θ_E and roll ϕ_E . The base rotation $U_A R$ used for both loci is $F_R(-1.2, 0.2, -1.3, -1)$. The non-circularity of the Euler locus, as well as the non-uniformity of the associated keypoints, demonstrate the violation of axisymmetry for Euler pitch and roll.

mathematical expression that epitomises the axisymmetry of fused pitch and roll, in the sense that they vary in a rotational manner proportional to the choice of β . The effect of varying β , and how this leads to a uniform circular locus of sine ratios $(\sin \phi_\beta, \sin \theta_\beta)$, is illustrated in Fig. 7. From (38), the phase shift to the quadrature sinusoid components can be seen to be $-\beta$. In consideration of (31), this yields the relation

$$\gamma_\beta = \gamma_0 - \beta. \quad (39)$$

This is an expression of the axisymmetry of the tilt axis angle γ , equivalent to that for fused pitch and roll. As (γ, α) completely parameterises the tilt rotation space (θ, ϕ, h) , it can be seen from (36) and (39) that all possible loci of sine ratios $(\sin \phi_\beta, \sin \theta_\beta)$ as β varies can be plotted by examining the contours of constant α while γ varies. This is equivalent to generating the level sets of constant $\sin \alpha$ in the fused pitch ratio vs. fused roll ratio plane, the result of which is shown in Fig. 8. The axisymmetry of fused pitch and roll can be clearly visually identified in the figure. An analogous plot for Euler angles is provided in Fig. 9. The non-axisymmetry of the Euler pitch and roll is clearly visible.

The non-axisymmetry of all three Euler angles parameters is further visualised in Fig. 10 and Fig. 11. The two figures also illustrate the corresponding axisymmetry of the fused angles parameters for the same base rotation $U_A R$. Conceptually, the problem of Euler pitch and roll is the fundamental requirement of a defined order of rotations. As can be identified in Fig. 5, this leads to definitions of pitch and roll that do not correspond to each other in behaviour, as one then implicitly depends on the value of the other.

VII. CONCLUSION

As has been shown in detail, fused angles possess many important properties that Euler angles do not. These proper-

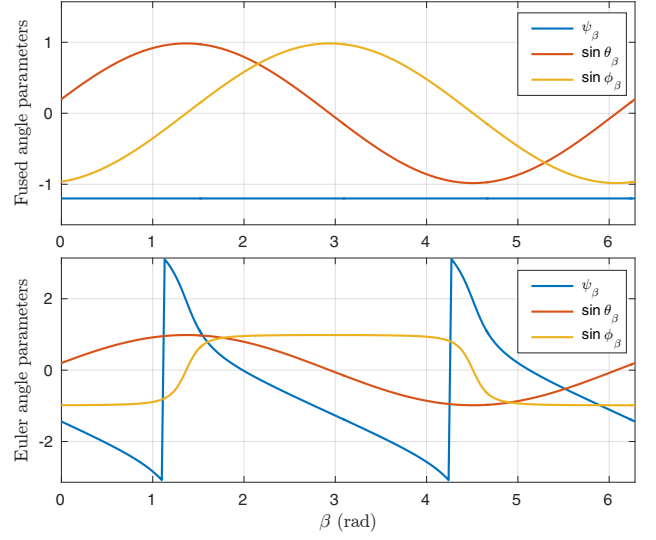


Fig. 11. Plots of yaw, and the pitch and roll sine ratios, against β for both fused angles (top) and Euler angles (bottom). The base rotation $U_A R$ used for both plots is $F_R(-1.2, 0.2, -1.3, -1)$, as in Fig. 10. The invariance of the fused yaw, as well as the exact quadrature nature of the fused pitch and roll can be clearly identified. The non-axisymmetry of Euler pitch and roll, and in particular the irregularity of Euler yaw, can also be seen.

ties relate to the nature of the singularities, the axisymmetry of the parameters, and the absence of any parameter inter-dependencies. As a result, fused angles not only intuitively quantify the amount of rotation within the three major planes, as was the core objective, but also fulfil natural and intuitive expectations about how a rotation representation should behave, especially for the application of representing the orientation of a balancing body [2].

REFERENCES

- [1] P. Allgeuer and S. Behnke, "Fused Angles: A representation of body orientation for balance," in *Int. Conf. on Intelligent Robots and Systems (IROS)*, Hamburg, Germany, 2015.
- [2] P. Allgeuer and S. Behnke, "Omnidirectional bipedal walking with direct fused angle feedback mechanisms," in *16th Int. Conf. on Humanoid Robots (Humanoids)*, Cancún, Mexico, 2016.
- [3] P. Allgeuer and S. Behnke, "Robust sensor fusion for biped robot attitude estimation," in *14th IEEE-RAS Int. Conference on Humanoid Robotics (Humanoids)*, Madrid, Spain, 2014.
- [4] H. Farazi, P. Allgeuer, G. Ficht, and S. Behnke, "NimbRo TeenSize team description 2016," in *RoboCup Humanoid League Team Descriptions*, 2016.
- [5] P. Allgeuer. (2018, Jan) Rotations Conversion Library. [Online]. Available: https://github.com/AIS-Bonn/rot_conv_lib
- [6] P. Allgeuer. (2018, Jan) Matlab/Octave Rotations Library. [Online]. Available: https://github.com/AIS-Bonn/matlab_octave_rotations_lib
- [7] B. Palais, R. Palais, and S. Rodi, "A disorienting look at Euler's theorem on the axis of a rotation," *The American Mathematical Monthly*, pp. 892–909, 2009.
- [8] S. Tomažič and S. Stančin, "Simultaneous orthogonal rotations angle," *Electrotechnical Review*, no. 78, pp. 7–11, 2011.
- [9] O. Bauchau and L. Trainelli, "The vectorial parameterization of rotation," *Nonlinear Dynamics*, vol. 32, no. 1, pp. 71–92, 2003.
- [10] L. Trainelli and A. Croce, "A comprehensive view of rotation parameterization," in *Proceedings of ECCOMAS*, 2004.
- [11] J. Stuelpnagel, "On the parametrization of the three-dimensional rotation group," *SIAM Review*, vol. 6, no. 4, 1964.
- [12] B. Palais and R. Palais, "Euler's fixed point theorem: The axis of a rotation," *J. of Fixed Point Theory and App.*, vol. 2, no. 2, 2007.